Motion Coordination with Noisy Measurement in Natural and Artifical Swarms

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***Abstract*— We consider the problem of controlling a group of mobile agents toward a formation defined by the desired relative positions between the agents. Each agent has available for control noisy measurements of its relative position with respect to a small set of neighbors. The motion of the group as a whole is due to a leader who moves independently of the other agents. We show that there are intrinsic limitations on the size of the group group determined by the underlying network structure imposed by the requirement of local interaction, which determine how low a tracking error can be achieved.**

It is shown that the tracking error covariance is given by the matrix-valued effective resistance introduced by the authors in Barooah and Hespanha (2005). We show how the effective resistance of a node in the multi-agent graph scales with the distance of that node from the leader for a large class of graphs. These scaling laws ultimately dictate on what kind of graphs scalable motion coordination can be achieved. Apart from providing design guidelines for robotic swarms, these results shed light on the dynamics of collective motion of certain animal groups.

I. Introduction

Distributed control algorithms for motion coordination of multi-agent autonomous systems have attracted considerable attention in recent times due to the promise of autonomous agents to perform a wide range of tasks that make them attractive to both military and civilian applications (Schoenwald, 2000). Usually an agent can interact with only a small subset of other agents. The constraints on the interactions between nodes define a graph whose nodes/vertices are the agents and the edges connect interacting agents. Many biological systems (e.g., bird flocks and fish schools) can also be viewed as multi-agent systems and their performance is limited by the same constraints that artificial multi-agent systems have to face due to limited and unreliable communication.

We consider the problem of controlling a group of mobile agents towards a formation defined by the desired relative positions between the agents. Each agent has available for control noisy measurements of its relative position with respect to a small set of neighbors. Noise in the measurements is inevitable in practice, and its affect on formation is a focus of our work. The group moves as a whole due to the motion of a leader, who moves independently of other agents. This problem is relevant for motion coordination of mobile autonomous agents where the common goal of the group may require that the agents form a specific formation. It is also relevant in the study of natural swarming behavior of animals; in particular, how specific formations are maintained using only local information. Although swarming in nature has been studied extensively (see (Okubo, 1986) and references therein), the effect of formation structure and size in the presence of noise has not been fully understood.

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Observation of biological multi-agent systems suggests that there may be intrinsic limitations on the size of the group determined by the underlying network structure, which in turn is determined by the requirement of local interaction between agents. For example, the V-formations used by flocks of geese (supposedly to reduce drag or improve lift) have far fewer individuals than the 3-dimensional formations that schools of herrings use. In this paper we show how the structure of the network imposes fundamental limitations on what objective the network of agents can collectively achieve when constrained to local interaction and noisy measurements.

To reveal these intrinsic limitations, we consider a distributed control law based on local noisy relative position measurements. The control law captures, in a simplified way, features present in the the more sophisticated models of animal swarming (Okubo, 1986). We examine the tracking error of individual agents depend on the size and structure of the network formed by the agents. We show that the tracking error covariance of an agent is equal to the matrix-valued effective resistance (Barooah and Hespanha, 2005) between the agent and the leader. We establish several properties of effective resistance. In particular, (i) we show that the effective resistances are monotone with respect to an appropriately defined graph embedding relation, and (ii) we determine laws that characterize how effective resistances scale with the distance between nodes. The scaling laws depends on the "denseness" and "sparseness" properties of the graph, and different graphs may exhibit very different scaling laws. Some graphs exhibit high rates of growth of the effective resistance, whereas in certain graphs effective resistance is bounded irrespective of how large the graph is. As a result, there are formation structures where it is possible to have low tracking error even with a large number of agents whereas in others it is fundamentally impossible to do so. These results can be used to explain some of the observed differences in the swarming behavior of different groups of animals, and also as guidelines in the design of robotic formations that allow for accurate tracking.

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II. Graph effective resistances

In this section we formally define the concept of matrix-valued effective resistance introduced by the authors in (Barooah and Hespanha, 2005) and refined in (Barooah and Hespanha, 2006b). An *undirected matrix-weighted graph* is a triple $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$, where $\mathbf{V}$ is a set of n vertices; $\mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ a set of m edges; and $\mathbf{W} := \{W_{u,v} \in \mathbb{R}^{k \times k} : (u, v) \in \mathbf{E}\}$ a set of symmetric positive definite matrix-valued weights for the edges of $\mathbf{G}$. Since we are dealing with undirected graphs, the pairs (u, v) and (v, u) denote the same edge. For simplicity of notation, we exclude the existence of multiple edges between the same pair of nodes and also edges from a node to itself.

The *matrix-weighted Laplacian* of $\mathbf{G}$ is a $nk \times nk$ matrix L with k rows per vertex and k columns per edge such that the $k \times k$ block of L corresponding to the k rows associated with node $u \in \mathbf{V}$ and the k columns associated with node $v \in \mathbf{V}$ is equal to

$$\begin{cases} \sum_{v \in \mathcal{N}_u} W_{u,v} & u = v \\ -W_{u,v} & (u, v) \in \mathbf{E} \\ 0 & (u, v) \notin \mathbf{E}. \end{cases}$$

where $\mathcal{N}_u \subset \mathbf{V}$ denotes the set of neighbors of u, i.e., the set of nodes that have an edge in common with u.

The matrix-weighted *Dirichlet or Grounded Laplacian* is obtained from the Laplacian by removing particular rows and columns. In particular, given a subset $\mathbf{V}_o \subset \mathbf{V}$ consisting of $n_o < n$ nodes, the *matrix-weighted Dirichlet Laplacian for the boundary* $\mathbf{V}_o$ is a $(n - n_o)k \times (n - n_o)k$ matrix L_o obtained from the matrix-weighted Laplacian of $\mathbf{G}$ by removing all rows and columns of L_o corresponding to the nodes in $\mathbf{V}_o$.

The usual graph Laplacian is a special case of the matrix-weighted Laplacian when $k = 1$ and all the weights are equal to one. The submatrix L_o of L in the special case of $k = 1$ is called the Dirichlet Laplacian, since it arises in the numerical solution of PDE's with Dirichlet boundary conditions (Chung, 1997). It is also called a Grounded Laplacian since it arises in the solution of node potentials in a electrical network (Ayazifar, 2002).

We say that a graph $\mathbf{G}$ is *connected to* $\mathbf{V}_o$ if there is a path from every node in the graph to at least one of the boundary nodes in $\mathbf{V}_o$. Lemma 1 at the end of this section shows that the Dirichlet Laplacian L_o is invertible when $\mathbf{G}$ is connected to $\mathbf{V}_o$.

We now formally define effective resistance in a connected graph $\mathbf{G}$: *node u's effective resistance to* $\mathbf{V}_o$, denoted by $R_u^{\text{eff}}(\mathbf{V}_o)$, is the $k \times k$ block in the main diagonal of L_o^{-1} corresponding to the k rows/columns associated with the node $u \in \mathbf{V}$. This terminology is justified by the fact that these matrices also express a map from (matrix-valued) currents to (matrix-valued) voltages in an appropriately defined electrical network. To make this connection precise, consider an abstract *generalized electrical network* where currents, potential drops and edge-resistances are $k \times k$ matrices. For such networks, Kirchoff's current law can be defined in the usual way, except that currents are added as matrices. Kirchoff's voltage law can also be defined in the usual way where potentials drops across edges are added as matrices. Kirchoff's voltage laws show the existence of a matrix valued node potential function. Ohm's law takes the form $V_e = R_e i_e$, where i_e is a generalized $k \times k$ matrix current flowing through the edge e of the electrical network, R_e is the generalized resistance of that edge, and V_e is a generalized $k \times k$ matrix potential drop across the edge e. Generalized resistances are always symmetric positive definite matrices.

The generalized electrical networks so defined share many of the properties of "regular" electrical networks. In particular, Kirchoff's and Ohm's laws uniquely define all edge currents and node voltages in a generalize electrical network when the potential at a particular *reference* node is fixed at some arbitrary value ("grounded"). The potential difference between pairs of nodes are the same irrespective of what the potential of the reference node is. This allow us to define the *generalized effective resistance* $R_{u,v}^{\text{eff}}$ between two nodes u and v as the potential difference between them when a generalized current equal to the $k \times k$ identity matrix is injected into one and extracted at the other.

The following result in (Barooah and Hespanha, 2005) justifies the terminology of graph effective resistances in section II by showing that it is the same as the generalized effective resistance defined above.

Theorem 1: Consider an undirected matrix-weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ and construct a generalized electric network with $k \times k$ edge-resistors R_e, $e \in \mathbf{E}$ that are numerically equal to the inverses of the corresponding edge-weights, i.e., $R_e = W_e^{-1}$, $e \in \mathbf{E}$. For every single-node boundary $\mathbf{V}_o := \{o\}$ and every node $u \in \mathbf{V} \setminus \mathbf{V}_o$, node u's *effective resistance to* $\mathbf{V}_o$ is equal to the *generalized effective resistance between u and o.* □

It should be noted that this effective resistance is matrix-valued. The previous definitions relied on the non-singularity of L_o, which is established by the following lemma (Barooah and Hespanha, 2006a).

Lemma 1 (Invertibility): The matrices L_o and L are both positive semi-definite. Moreover, the matrix L_o is positive definite if and only if $\mathbf{G}$ is connected to $\mathbf{V}_o$. □

We will shorty see that effective resistances play a key role in scalable motion coordination. Moreover, effective resistances also allow us to deduce properties of the spectrum of the matrix-weighted Dirichlet Laplacian and even of the spectrum of the original matrix-weighted Laplacian. The convergence of several continuous-time distributed algorithms is determined by the spectrum of the matrix

$$G_o := -\gamma D_o^{-1} L_o, \quad \gamma > 0.$$

where D_o is a block diagonal matrix that contains the $k \times k$ blocks on the diagonal of the matrix-weighted Dirichlet Laplacian L_o, and γ is a positive constant. The following lemma that was proved in (Barooah and Hespanha, 2006a) will be useful in later analysis.

Lemma 2 ((Barooah and Hespanha, 2006a)): Assume that $\mathbf{G}$ is connected to $\mathbf{V}_o$. Every eigenvalue of G_o is real and satisfies

$$\lambda_i(G_o) \leq -\frac{\gamma}{\lambda_{\max}(D_o) \sum_{u \in \mathbf{V}} \operatorname{trace} R_u^{\text{eff}}(\mathbf{V}_o)}, \qquad (1)$$

where $\lambda_{\max}(D_o)$ denotes the largest eigenvalue of D_o. □

III. Formation control with noisy measurements

Consider a group of n mobile agents moving in k-dimensional space that one desires to control to a given formation defined by their relative positions. In particular, denoting by $x_u \in \mathbb{R}^k$, $u \in \mathbf{V} := \{1, 2, \ldots, n\}$ the position of the uth agent, the control objective is to make the positions converge to values for which

$$x_u - x_v = r_{u,v}, \quad \forall (u, v) \in \mathbf{V} \times \mathbf{V}, \tag{2}$$

where $r_{u,v}$ denotes the desired relative position of agent u with respect to agent v. One of the agents $o \in \mathbf{V}$ will be called the *leader* and it will move independently of the remaining ones. The remaining agents attempt to maintain the formation specified by (2). The leader may actually not be a physical agent. Instead, it may be a "reference" that is known to at least one of the physical agents.

Not all agents are able to measure their relative positions with respect to all other agents and therefore each agent is constrained to use only a few relative position measurements to compute its control signal. We denote by $\mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ the set of pairs of agents that can measure their relative positions. In particular, the existence of a pair (u, v) in $\mathbf{E}$ signifies that agent u can measure its position with respect to v and similarly, v can measure its position with respect to u, although both measurements will be corrupted with noise. Since the noise corrupting the measurement of $x_u - x_v$ available to u will be in general different from the noise on the measurement of $x_v - x_u$ available to v, we need to distinguish these two measurements. To this end, we introduce a *directed* edge set $\vec{\mathbf{E}}$ containing the two ordered pairs $(u, v), (v, u)$ whenever $(u, v) \in \mathbf{E}$. We assume that a noisy measurement $y_{u,v}$ of the following form is available to agent u if $(u, v) \in \vec{\mathbf{E}}$:

$$y_{u,v} = x_u - x_v + \epsilon_{u,v} \tag{3}$$

where $\epsilon_{u,v}$ is a white random noise process with autocorrelation matrix given by $\mathrm{E}[\epsilon_{u,v}(t_1)\epsilon_{u,v}^T(t_2)] = \delta(t_1 - t_2)R_{u,v}$.

Note that by assumption, if a measurement $y_{u,v}$ is available to u, then the measurement $y_{v,u}$ is available to v. The noise processes over different edges are assumed independent of each other. In particular, $e_{u,v}(t)$ is independent of $e_{v,u}(t)$ for all t. In case x_o is a reference and not a physical agent, an edge between the node u and the leader o means that the physical agent u is able to measure its position with respect to the reference o.

The problem above is now associated with a matrix-weighted *directed* graph $\vec{\mathbf{G}} = (\mathbf{V}, \vec{\mathbf{E}}, \vec{\mathbf{W}})$ with node set $\mathbf{V} = \{1, 2, \ldots, n\}$; directed edge set $\vec{\mathbf{E}}$ consisting of all ordered pairs of nodes (u, v) for which a noisy measurement of the form (3) is available; and weight set $\vec{\mathbf{W}}$ consisting of the inverses of the autocorrelation matrices $W_{u,v} := R_{u,v}^{-1}$, $(u, v) \in \vec{\mathbf{E}}$. We assume that even though the measurement errors on the two edges (u, v) and (v, u) connecting the nodes u and v are independent, they have the same autocorrelation matrix; i.e., $R_{u,v} = R_{v,u}$. We will refer to this assumption, together with the assumption that the directed edge (u, v) exists iff (v, u) exists, as *bidirectionality*. Fig. 1 shows an example of a bidirectional directed graph and its associated undirected graph.

We are interested in control laws for which each agent uses all its measurements to construct an optimal estimate of the difference between its currently position and what this "should" be, in view of what it know about its neighbors positions. The measurements available to an arbitrary agent $u \in \mathbf{V}$ are

$$y_{u,v} = x_u - x_v + \epsilon_{u,v}, \quad \forall v \in \mathcal{N}_u,$$

where $\mathcal{N}_u \subset \mathbf{V}$ denotes set of nodes v such that $(u, v) \in \vec{\mathbf{E}}$. If agent u assumes that all its neighbors are correctly positioned then, according to (2), the desired position of u is given by any one of the following equations

$$x_u^d = x_v + r_{u,v}, \quad \forall v \in \mathcal{N}_u.$$

Combining the two previous sets of equations, we obtain

$$y_{u,v} = x_u - x_u^d + r_{u,v} + \epsilon_{u,v}, \quad \forall v \in \mathcal{N}_u,$$

from which agents u estimates its position error $x_u - x_u^d$. It is straightforward to show that the Best Linear Unbiased estimate of $x_u - x_u^d$ is given by

$$D_u^{-1} \sum_{v \in \mathcal{N}_u} R_{u,v}^{-1}\big(y_{u,v} - r_{u,v}\big),$$

where $D_u := \sum_{v \in \mathcal{N}_u} R_{u,v}^{-1}$. This motivates the following negative proportional control law for the agents

$$\dot{x}_u = -\gamma D_u^{-1} \sum_{v \in \mathcal{N}_u} R_{u,v}^{-1}(y_{u,v} - r_{u,v}), \quad \forall u \in \mathbf{V} \setminus \{o\}, \tag{4}$$

where γ denotes some positive number. For analysis purposes it is convenient to describe the system dynamics in term of positions with respect to the leader. Defining $\tilde{x}_u = x_u - x_o$, one concludes that

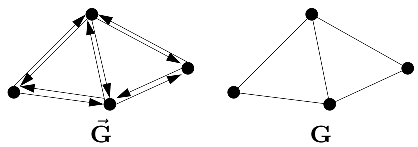
$$\dot{\tilde{x}}_u = -\gamma D_u^{-1} \sum_{v \in \mathcal{N}_u} R_{u,v}^{-1}(\tilde{x}_u - \tilde{x}_v - r_{u,v} + \epsilon_{u,v}) - \dot{x}_o,$$

$\forall u \in \mathbf{V} \setminus \{o\}$.

By stacking all the positions $\tilde{x}_u$, $u \in \mathbf{V} \setminus \{o\}$ in a column vector $\tilde{x}$, the above systems can be written as follows:

$$\dot{\tilde{x}} = -\gamma D_o^{-1} L_o \tilde{x} + \gamma D_o^{-1} B_o W(r - \epsilon) - \dot{x}_o \mathbf{1}, \tag{5}$$

where r is a column vector obtained by stacking all the $r_{u,v}$ on top of each other; ϵ is a column vector obtained by stacking all the $\epsilon_{u,v}$; $\mathbf{1}$ is a $n-1 \times 1$ column vector of all 1's; $W > 0$ is a block-diagonal matrix with k rows/columns for each edge in $\vec{\mathbf{E}}$, with the weights $W_{u,v} := R_{u,v}^{-1}$, $(u, v) \in \vec{\mathbf{E}}$ in the diagonal; $D_o > 0$ is a block-diagonal matrix with k rows/columns for each node in $\mathbf{V} \setminus \mathbf{V}_o$, with D_u, $u \in \mathbf{V} \setminus \mathbf{V}_o$ as defined earlier in the diagonal; $L_o = \frac{1}{2}\mathcal{A}_o W \mathcal{A}_o^T$ where $\mathcal{A}_o$ is the generalized incidence matrix for the directed graph $(\mathbf{V}, \vec{\mathbf{E}})$ with $V_o = \{o\}$ (cf. section II); and B_o is a matrix with k rows for each vertex in $\mathbf{V} \setminus \mathbf{V}_o$ and k columns for each edge in $\vec{\mathbf{E}}$, constructed as follows: the k columns corresponding to edge $(u, v) \in \vec{\mathbf{E}}$ are all equal to zero except for the block corresponding to the node u, which is equal to I_k. The white noise process ϵ has block diagonal autocorrelation



$$\mathcal{A}_0 = \begin{bmatrix} -1 & 0 & I & I & I & 0 & -I & 0 & -I & 0 \\ 0 & -I & 0 & -I & 0 & I & I & I & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & -I & I & I \end{bmatrix} \quad B_o = \begin{bmatrix} I & I & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & I & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & I \end{bmatrix}$$

$$L = \left[\begin{array}{c|ccc} 2I & -2I & 0 & 0 \\ \hline -2I & 4I & -I & -I \\ 0 & -I & 2I & -I \\ 0 & \multicolumn{3}{c}{\underbrace{-I \quad -I \quad 2I}_{L_o}} \end{array}\right], \quad D_o = \begin{bmatrix} 4I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 2I \end{bmatrix}$$

Fig. 1. A bidirectional directed graph $\vec{\mathbf{G}}$ and the associated undirected graph $\mathbf{G}$. The matrices $\mathcal{A}_o$ and B_o shown are for the graph $\vec{\mathbf{G}}$.

matrix given by $\mathrm{E}[\epsilon(t_1)\epsilon^T(t_2)] = \delta(t_1-t_2)W^{-1}$. Fig. 1 shows an example of the matrices defined above.

L_o is exactly the matrix-weighted Dirichlet Laplacian for the matrix-weighted undirected graph $(\mathbf{V}, \mathbf{E}, \mathbf{W})$ with boundary $\mathbf{V}_o := \{o\}$ with the weight $W_{u,v}$ on every undirected edge (u, v) assigned as the weight on the corresponding directed edge $(u, v) \in \vec{\mathbf{E}}$. Note that we get the undirected Laplacian in the system dynamic equations (5) due to the bidirectionality assumption.

From Lemma 2, we conclude that (5) is an asymptotically stable system and Lemma 2 actually relates the speed of its slowest pole with the effective resistances. It turns out that the effective resistances play an even more interesting role for this system. To see this, we further re-write the model (5) as

$$\dot{\tilde{x}} = -\gamma D_o^{-1} L_o \tilde{x} + w + b,$$

where $b := \gamma D_o^{-1} B_o W r - \dot{x}_o \mathbf{1}$ and $w := -\gamma D_o^{-1} B_o W \epsilon$ is a white noise random process with autocorrelation matrix given by

$$\begin{aligned} \mathrm{E}[w(t_1) w^T(t_2)] &= \gamma^2 D_o^{-1} B_o W \,\mathrm{E}[\epsilon(t_1)\epsilon^T(t_2)] W B_o^T D_o^{-1} \\ &= \gamma^2 \delta(t_1 - t_2) D_o^{-1} B_o W B_o^T D_o^{-1} = \gamma^2 \delta(t_1 - t_2) D_o^{-1}, \end{aligned}$$

where we used the fact that $B_o W B_o^T = D_o$. Since the Lyapunov equation

$$-\gamma D_o^{-1} L_o \Sigma_\infty - \gamma \Sigma_\infty L_o D_o^{-1} + \gamma^2 D_o^{-1} = 0$$

has a positive definite solution

$$\Sigma_\infty = \frac{\gamma}{2} L_o^{-1},$$

it is straightforward to show that the covariance matrix of $\tilde{x}$ converges to Σ_∞. In particular, the steady-state covariance matrix of the relative position $\tilde{x}_u := x_u - x_o$ is given by $k \times k$ diagonal block of Σ_∞, which is given by $\gamma/2$ times node u's effective resistance $R^{\text{eff}}_{u,o}$ to $\mathbf{V}_o := \{o\}$ defined in Section II.

The above analysis of the closed loop tracking error dynamics shows that with the control law (4), the tracking error covariance of an agent turns out to be equal to its effective resistance with the leader. In (Barooah and Hespanha, 2006b), upper and lower bounds were established for the scaling law of effective resistance as a function of distance from the leader. In the subsequent sections we summarize these scaling laws. In the remaining part of this section, we discuss several well known results for "regular" electrical networks that can be adapted to generalized electrical networks.

A. Rayleigh's Monotonicity Law

Rayleigh's Monotonicity Law (Doyle and Snell, 1984) states that if the edge-resistances in a (regular) electrical network are increased, then the effective resistance between any two nodes in the network can only increase. Conversely, a decrease in edge-resistances can only lead to a decrease in effective resistance. It turns out that Rayleigh's Monotonicity Law can be extended to generalized electrical networks.

For the problems considered here, it is convenient to consider not only increases in edge-resistances but also removing an edge altogether or introducing a new edge. The statement of a version of Rayleigh's Monotonicity Law that allow us to consider edge removal requires the introduction of a partial order for graphs. Given two undirected matrix-weighted graphs

$$\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W}), \qquad \bar{\mathbf{G}} = (\bar{\mathbf{V}}, \bar{\mathbf{E}}, \bar{\mathbf{W}})$$

we say that $\mathbf{G}$ *can be embedded in* $\bar{\mathbf{G}}$, and write $\mathbf{G} \subset \bar{\mathbf{G}}$, if $\mathbf{V} \subset \bar{\mathbf{V}}$, $\mathbf{E} \subset \bar{\mathbf{E}}$, and

$$W_{u,v} \leq \bar{W}_{u,v}, \quad \forall (u, v) \in \mathbf{V}.$$

Here and below, given two symmetric matrices A and B, we write $A \geq B$ to mean that the matrix $A - B$ is positive semi-definite.

It was proved in (Barooah and Hespanha, 2006b) that Rayleigh's monotonicity law holds for generalized electrical networks. In view of Theorem 1, this leads to the following monotonicity result for graph effective resistances:

Theorem 2 (Rayleigh's Generalized Monotonicity Law): Consider two undirected matrix-weighted graphs

$$\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W}), \qquad \bar{\mathbf{G}} = (\bar{\mathbf{V}}, \bar{\mathbf{E}}, \bar{\mathbf{W}})$$

such that $\mathbf{G} \subset \bar{\mathbf{G}}$. For every single-node boundary $\mathbf{V}_o := \{o\} \in \mathbf{V}$ and every node $u \in \mathbf{V} \setminus \mathbf{V}_o$, we have that

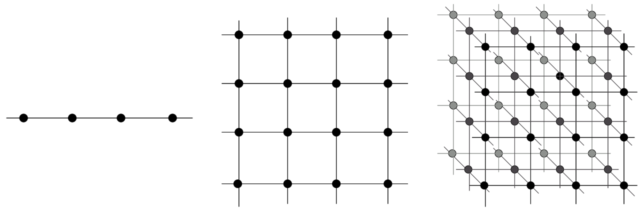
$$R^{\text{eff}}_u(\mathbf{V}_o) \geq \bar{R}^{\text{eff}}_{\bar{u}}(\mathbf{V}_o),$$

where $R^{\text{eff}}_u(\mathbf{V}_o)$ denotes u's effective resistance to $\mathbf{V}_o$ with respect to the graph $\mathbf{G}$ and $\bar{R}^{\text{eff}}_{\bar{u}}(\mathbf{V}_o)$ denotes u's effective resistance to $\mathbf{V}_o$ with respect to the graph $\bar{\mathbf{G}}$. □

B. Lattices, h-fuzzes, and their effective resistance

The effective resistances of several "regular" electrical networks have been studied in the literature on resistive electrical network. In this section we discuss a few graphs for which results can also be obtained for generalized electric networks.

A *d-dimensional lattice*, denoted by $\mathbb{Z}_d$ is a graph that has one vertex for every point in $\mathbb{R}^d$ with integer coordinates and an edge between every two vertices corresponding to points with an Euclidean distance between them equal to one.



(a) 1-dimensional lattice $\mathbb{Z}_1$ (b) 2-dimensional lattice $\mathbb{Z}_2$ (c) 3-dimensional lattice $\mathbb{Z}_3$

Fig. 2. Lattices

Figure 2 shows 1-, 2-, and 3-dimensional lattices. Lattices have infinitely many nodes and edges, and are therefore examples of *infinite graphs*. In practice, they serve as proxies for very large graphs.

We now need to define the concept of a fuzz of a graph, for which we introduce the notion of graphical distance. Given two nodes u and v of a graph $\mathbf{G}$, their *graphical distance*, denoted by $d_{\mathbf{G}}(u, v)$ is the minimum number of edges one has to traverse in going from one node to the other.

Given a graph $\mathbf{G}$ and an integer $h \geq 1$, *h-fuzz of* $\mathbf{G}$, denoted by $\mathbf{G}^{(h)}$, is a graph with the same set of nodes as $\mathbf{G}$ but with a larger set of edges. In particular, $\mathbf{G}^{(h)}$ has an edge between u and v whenever the graphical distance between u and v is less than or equal to h (Doyle and Snell, 1984). Figure 3 shows the 2-fuzz of the 2-dimensional lattice.

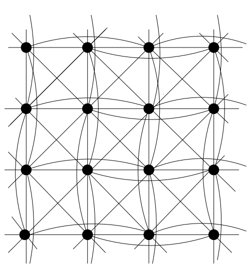


Fig. 3. 2-fuzz of the 2-dimensional lattice.

It was shown by (Doyle, 1998) that the scalar effective resistance in $\mathbf{G}^{(h)}$ (with all edges in both $\mathbf{G}^{(h)}$ and $\mathbf{G}$ having 1-Ohm resistors) will be lower than the corresponding effective resistance in $\mathbf{G}$ *only by a constant factor* as long as h is finite, and that constant will depend on h but not on the graphical distance between the two nodes. The arguments in (Doyle, 1998) can be used to show that the same result holds for generalized electrical networks, which is stated in the next lemma (see (Barooah and Hespanha, 2006b) for details).

Lemma 3: Let $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W}_o)$ and $\mathbf{G}^{(h)} = (\mathbf{V}, \mathbf{E}^{(h)}, \mathbf{W}_o)$ be two matrix-weighted graphs where the graphs $\mathbf{G}$ and $\mathbf{G}^{(h)}$ have bounded degree and all edges in both the graphs have equal positive definite weight W_o. Let $R^{\text{eff}}_{u,v}(\mathbf{G})$ be the effective resistance between two nodes u and v in $\mathbf{G}$ and $R^{\text{eff}}_{u,v}(\mathbf{G}^{(h)})$ be the effective resistance between u and v in the h-fuzz $\mathbf{G}^{(h)}$. The following relationship holds:

$$\alpha R^{\text{eff}}_{u,v}(\mathbf{G}) \leq R^{\text{eff}}_{u,v}(\mathbf{G}^{(h)}) \leq R^{\text{eff}}_{u,v}(\mathbf{G}),$$

where $\alpha \in (0, 1]$ is a constant independent of u and v.

The next lemma from (Barooah and Hespanha, 2005) establishes the effective resistances in d-dimensional lattices and their fuzzes.

Lemma 4 (Lattice Generalized Effective Resistances): Consider a generalized electrical network obtained by placing generalized matrix resistances equal to R_o at the edges of the h-fuzz of the d-dimensional lattice, where h is a positive integer, $d \in \{1, 2, 3\}$, and R_o is a symmetric positive definite $k \times k$ matrix. There exist constants $\ell, \alpha_i, \beta_i > 0$ such that the formulas in Table I hold for every pair of nodes u, v at a graphical distance larger than ℓ. $\square$

The fact that in a 1-dimensional lattice the effective resistance grows linearly with the distance between nodes can be trivially deduced from the well known formula for the effective resistance of a series of resistors (which extends to generalized electrical networks). In two-dimensional lattices the effective resistance only grows with the logarithm of the graphical distance and therefore the effective resistance grows very slowly with the distance between nodes. Far more surprising is the fact that in three-dimensional latices the effective resistance is actually bounded by a constant even when the distance is arbitrarily large.

IV. Scaling of Effective Resistance with Distance: Dense and Sparse Graphs

In this section we show how to combine the Electrical Analogy Theorem 1, Rayleigh's Generalized Monotonicity Law, and the Lattice Effective Resistance Lemma 4 to determine scaling laws of the effective resistance for more general classes of graphs.

A higher density of edges and nodes in a graph should lead to lower effective resistances. However, "naive" measures of node and edge density turn out to be misleading predictors for how the effective resistance scales with distance. We shall see that to predict these scalability laws one needs instead to determine deeper structural properties of the graph. First, however, we will discuss a few examples that will motivate our results.

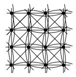
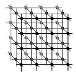
A. Graph Drawing

In graph theory, a graph is generally treated purely as a collection of nodes connected by edges, without any regard to the geometry determined by the nodes' locations. However, for the graphs that arise in distributed control problems there is an underlying geometry because nodes generally correspond to physical agents and their locations often determine the connectivity of the graph (Barooah and Hespanha, 2006a).

Graph drawings are used to capture the geometry of graphs in Euclidean space. The drawing of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ is simply a mapping of its nodes to points in some Euclidean space, which can formally be described by a function $f : \mathbf{V} \to \mathbb{R}^d,\ d \geq 1$. A drawing is also sometimes called a *representation* of a graph (Godsil and Royle, 2001). For a

TABLE I

EFFECTIVE RESISTANCE FOR LATTICES AND THEIR FUZZES

Graph		Effective resistance between u and v
	$\mathbb{Z}_1^{(h)}$	$\alpha_1 d_{\mathbb{Z}_1}(u,v) R_o \leq R_{u,v}^{\text{eff}} \leq \beta_1 d_{\mathbb{Z}_1}(u,v) R_o$
	$\mathbb{Z}_2^{(h)}$	$\alpha_2 \log\big(d_{\mathbb{Z}_2}(u,v)\big) R_o \leq R_{u,v}^{\text{eff}} \leq \beta_2 \log\big(d_{\mathbb{Z}_2}(u,v)\big) R_o$
	$\mathbb{Z}_3^{(h)}$	$\alpha_3 R_o \leq R_{u,v}^{\text{eff}} \leq \beta_3 R_o$

particular drawing f of a graph, we can define Euclidean distances between nodes, which are simply the distances in Euclidean space between the drawings of the nodes. In particular, given two nodes $u, v \in \mathbf{V}$ the *Euclidean distance between* u *and* v *induced by the drawing* $f : \mathbf{V} \to \mathbb{R}^d$ is defined by

$$d_f(u,v) := \|f(v) - f(u)\|,$$

where $\|\cdot\|$ denoted the usual Euclidean norm in d-space. Euclidean distances depend on the drawing and can be completely different from graphical distances. It is important to emphasize that the definition of drawing does not require edges to not intersect and therefore every graph has a drawing in any Euclidean space. In fact, every graph has infinitely many drawings.

For graphs that arise in distributed control problems in which nodes correspond to physical agents, there is a *natural drawing* that is obtained by associating each node to its position in 1-, 2- or 3-dimensional Euclidean space. In reality, all agents are situated in 3-dimensional space. However, sometimes it maybe more natural to draw them on a 2-dimensional Euclidean space if one dimension (e.g., height) does not vary much from node to node, or is somehow irrelevant. For *natural drawings, the Euclidean distance induced by the drawing is, in general, a much more meaningful notion of distance than the graphical distance.* In this paper we will see that the Euclidean distance induced by appropriate drawings provide the right measure of distance to determine scaling laws of effective resistance.

B. Measures of denseness/sparseness

For a particular drawing f and induced Euclidean distance d_f of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$, four parameters can be used to characterize denseness/sparseness. *Minimum node distance* denotes the minimum Euclidean distance between the drawing of any two nodes:

$$s := \inf_{\substack{u,v \in \mathbf{V} \\ v \neq u}} d_f(u,v).$$

Maximum connected range denotes the Euclidean length of the drawing of the longest edge:

$$r := \sup_{(u,v)\in\mathbf{E}} d_f(u,v).$$

Maximum uncovered diameter denotes the diameter of the largest open ball that can be placed in $\mathbb{R}^d$ with no drawing of a node inside it:

$$\gamma := \sup\Big\{\delta : \exists B_\delta \text{ such that } f(u) \notin B_\delta,\ \forall u \in \mathbf{V}\Big\},$$

where the existential quantification spans over the balls B_δ in $\mathbb{R}^d$ with diameter δ. Finally, *asymptotic distance scaling* denotes the largest asymptotic ratio between the graphical and the Euclidean distance between two nodes:

$$\rho := \lim_{n\to\infty} \inf \Big\{ \frac{d_f(u,v)}{d_{\mathbf{G}}(u,v)} : u, v \in \mathbf{V} \text{ and } d_{\mathbf{G}}(u,v) \geq n \Big\}.$$

Essentially ρ provides a lower bound for the ratio between the Euclidean and the graphical distance for nodes that are very far apart.

1) Dense graphs: The drawing of a graph for which the maximum uncovered diameter is finite ($\gamma < \infty$) and the asymptotic distance scaling is positive ($\rho > 0$) is called a *dense drawing*. We say that a $\mathbf{G}$ *is dense in* $\mathbb{R}^d$ if there exists a dense drawing of the graph in $\mathbb{R}^d$. Intuitively, these drawing are "dense" in the sense that the nodes can cover $\mathbb{R}^d$ without leaving large holes between them and still having sufficiently many edges so that a small Euclidean distance between two nodes in the drawing guarantees a small graphical distance between them. In particular, for dense drawings there are always finite constants α, β for which

$$d_{\mathbf{G}}(u,v) \leq \alpha\, d_f(u,v) + \beta, \quad \forall u, v \in \mathbf{V}.$$

This fact is proved in (Barooah and Hespanha, 2006b). Using the natural drawing of a d-dimensional lattice, one concludes that this graph is dense in $\mathbb{R}^d$. One can also show that a d-dimensional lattice can never be dense in $\mathbb{R}^{\bar{d}}$ with $\bar{d} > d$. This means, for example, that any drawing of a 2-dimensional lattice in the 3-dimensional Euclidean space will never be dense.

2) Sparse graphs: Graph drawings for which the minimum node distance is positive ($s > 0$) and the maximum connected range is finite ($r < \infty$) are called *civilized drawings*. This definition is essentially a refinement of the one given in (Doyle and Snell, 1984), with the quantities r and s made to assume precise values. Intuitively, these drawings are "sparse" in the sense that one can keep the edges with finite lengths, without cramping all nodes on top of each other. We say that a graph $\mathbf{G}$ *is sparse in* $\mathbb{R}^d$ if *it can be drawn in a civilized manner in* d*-dimensional Euclidean space*.

The notions of graph "sparseness" and "denseness" are mostly interesting for infinite graph, because every finite graph is sparse in all Euclidean spaces $\mathbb{R}^d$, $\forall d \geq 1$ and no finite graph can ever be dense in any Euclidean space $\mathbb{R}^d$, $\forall d \geq 1$. This is because any drawing of a finite graph that does not place nodes on top of each other will necessarily have a positive minimum node distance and a finite maximum connected range (from which sparse follows) and it is not possible to achieve a finite maximum uncovered diameter with a finite number of nodes (from which lack of denseness follows). However, in practice infinite graphs serve as proxies for very large graphs that, from the perspective of most nodes, "appear to extend in all directions as far as the eye can see." So conclusions drawn for sparse/dense infinite graphs hold for large graphs, at least far from the graph boundaries.

3) Sparseness, denseness, and embeddings: The notions of sparseness and denseness introduced above are useful because they provide a complete characterization for the classes of graphs that can embed or be embedded in lattices, for which the Lattice Effective Resistance Lemma 4 provides the precise scaling laws for the effective resistance.

Theorem 3 (Lattice Embedding): Let $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ be a graph without multiple edges between the same pair of nodes.

1) $\mathbf{G}$ is sparse in $\mathbb{R}^d$ *if and only if* $\mathbf{G}$ can be embedded in an h-fuzz of a d-dimensional lattice. More precisely,
$$\mathbf{G} \text{ is sparse in } \mathbb{R}^d \quad \Leftrightarrow \quad \exists h < \infty : \mathbf{G} \subset \mathbb{Z}_d^{(h)}$$
2) $\mathbf{G}$ is dense in $\mathbb{R}^d$ *if and only if* (i) the d-dimensional lattice can be embedded in an h-fuzz of $\mathbf{G}$ for some positive integer h and (ii) every node of $\mathbf{G}$ is at an uniformly bounded graphical distance from another node of $\mathbf{G}$ that is also a node of $\mathbb{Z}_d$. More precisely,
$$\begin{aligned}\mathbf{G} \text{ is dense in } \mathbb{R}^d \quad \Leftrightarrow \quad & \exists h, c < \infty : \mathbf{G}^{(h)} \supset \mathbb{Z}_d \\ & \&\ \forall u \in \mathbf{V}\ \exists \bar{u} \in \mathbf{V}_{\text{lat}}(\mathbf{G}) : d_{\mathbf{G}}(u, \bar{u}) \leq c,\end{aligned}$$
where $\mathbf{V}_{lat}(\mathbf{G})$ denotes the nodes of $\mathbf{G}$ that are mapped to nods in $\mathbb{Z}_d$. □

The proof follows from simple geometric arguments, the interested reader is referred to (Barooah and Hespanha, 2006b) for the details.

C. Scaling laws for effective resistance

We are now finally ready to characterize scaling laws for graph effective resistances in terms of the denseness/sparseness properties of the graph. The following theorem does precisely this by combining Theorem 1, Rayleigh's Generalized Monotonicity Law, the Lattice Effective Resistance Lemma 4, and the Lattice Embedding Theorem 3.

Theorem 4 (Scaling of effective resistance): Consider an undirected matrix-weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ with matrix weights that satisfy $R_{\min} \leq W_e^{-1} \leq R_{\max}$, $\forall e \in \mathbf{E}$ for some symmetric positive definite matrices $R_{\min}$, $R_{\max}$. There exist constants $\ell, \alpha_i, \beta_i > 0$ such that the formulas in Table II hold for every single-node boundary $\mathbf{V}_o := \{o\}$ and every node u at an Euclidean to the boundary node o larger than ℓ. □

V. Implications for man-made autonomous agents

The results described in the previous section shows that for graphs that are dense in $1, 2$ and 3-dimensions, the effective resistance between a node and the leader is upper bounded by a linear, logarithmic and bounded function of the node's distance from the leader, respectively. On the other hand, if a graph is sparse in $1, 2$ and 3-dimensions, the effective resistance between a node and the leader is lower bounded by a linear, logarithmic and bounded function of the node's distance, respectively. Since tracking error covariance is exactly the same as the graph effective resistance, if a graph is sparse in 1-dimension, the tracking error variance increases linearly with distance, and so agents far away from the leader will have poor tracking error. On the other hand, if a graph is dense in 3-dimension, the tracking error variance will stay bounded by a constant no matter the distance. Clearly a group of mobile agents forming a 3-D dense graph can achieve accurate motion coordination even if the group has a large number of agents.

The preceding discussion shows that the maximum tracking errors in two networks consisting of the same number of agents can be quite different. As a result, some networks are more scalable than others in terms of tracking performance. This knowledge can be used for designing networks that are formed by groups of mobile autonomous agents, such as UAVs. Frequently, the formation structure of such agents is designed solely on the basis of the task that the group is expected to perform. However, our results show that a formation structure itself imposes fundamental limitations on how well that formation can be maintained by the agents. Thus, if the agents are required to maintain their formation accurately, then the desired formation itself has to be appropriately chosen. For example, it will be unwise to ask a large group of agents to fly in a single line while maintaining very accurate spacings between neighbors, since we know that in such a graph the tracking error grows linearly with the number of agents.

VI. Implications for Swarming in Nature

While the exact nature of motion coordination among biological agents is still a mystery, the control law (4) is nevertheless an approximation of the motion coordination schemes that are proposed to explain swarming behavior in animals (Okubo, 1986). This control law is extremely simple and requires only information about nearby agents, which can be obtained by vision and/or auditory sensing mechanisms. Moreover, measurement noise is likely to affect the relative position estimates as modelled in (4).

The tracking error variance resulting from such an algorithm can explain a number of puzzling observations from nature. For example, it is well know that many varieties of birds fly in a "V"-formation (cf. Fig. 4(a)). Although the explanation of why this happens is still a matter of debate (both drag reduction and better visual cue about positions have been offered (Cutts and Speakman, 1994) to explain this observation), it is observed that the birds close to the leader maintain relative positions quite well, while the birds toward the end of the arms of the "V" usually do not maintain well defined spacings with their neighbors. This might be explained by the fact that a V-formation is sparse in 1-dimension and hence the largest tracking error grows linearly with the number of birds in the flock. On the other extreme, schools consisting of millions of fish are known to move together in a 3-dimensional structure in a surprisingly agile fashion (Nottestad and Axelsen, 1999). In a large school of fish such as the one shown in Fig. 4(b), the network topology is 3-dimensional. It is not hard to see

TABLE II

EFFECTIVE RESISTANCES FOR GRAPHS THAT ARE SPARSE OR DENSE. IN THE TABLE, $d_f(u,o)$ DENOTES THE EUCLIDEAN DISTANCE BETWEEN NODE u AND THE REFERENCE NODE o, FOR ANY DRAWING f THAT ESTABLISHES THE GRAPH'S SPARSENESS/DENSENESS.

Euclidean space	Covariance matrix of the estimation error of x_u in a *sparse graph*	Covariance matrix of the estimation error of x_u in a *dense graph*
$\mathbb{R}$	$\alpha_1 d_f(u,o) R_{\min} \leq R^{\text{eff}}_{u,o}$	$R^{\text{eff}}_{u,o} \leq \beta_1 d_f(u,o) R_{\max}$
$\mathbb{R}^2$	$\alpha_2 \log\big(d_f(u,o)\big) R_{\min} \leq R^{\text{eff}}_{u,o}$	$R^{\text{eff}}_{u,o} \leq \beta_2 \log\big(d_f(u,o)\big) R_{\max}$
$\mathbb{R}^3$	$\alpha_3 R_{\min} \leq R^{\text{eff}}_{u,o}$	$R^{\text{eff}}_{u,o} \leq \beta_3 R_{\max}$

(a) A flock of birds in "V"-formation

(b) A school of fish

Fig. 4. Examples of 1-D and 3-D network topologies in natural swarms. Photograph in (b) courtesy Sergey Parinov (http://www.sergeyphoto.com)

that the network in such a large school will be dense in 3-D, and possibly can be approximated by a 3-dimensional lattice. Hence the tracking error variance of each agent remains bounded even when the number of agents (fish) making up the school is arbitrarily large. This might explain why large fish schools can move together and maneuver quickly even while forming an extremely large network while a comparatively small number of birds flying straight find it difficult to keep a constant separation.

VII. CONCLUSION

We considered the problem of motion coordination of a number of autonomous agents when each agent has access only to noisy, relative position measurements with its nearby neighbors. We showed that the covariance of the tracking error of an agent is equal to a matrix-valued effective resistance between the agent and the leader. Scaling laws for the effective resistance were established for graphs that satisfy appropriate "denseness/sparseness" properties. In graphs that are dense in 3-D, effective resistance of a node does not grow as the distance of a node from the leader increases, which makes these graphs highly scalable in terms of tracking performance. On the other hand, effective resistance grows linearly with distance in graph that are sparse in 1-D, so that the tracking performance is necessarily poor if the graph contains a large number of agents.

These results clarify the fundamental performance limitations that motion coordination of mobile autonomous agents suffer from. Design of robotic swarms should take into account these limitations in order to ensure that the design goal is achievable. In addition, our results also shed new light on the dynamics of animal swarms.

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